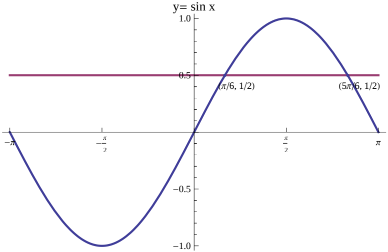
Restricted Sine Function.

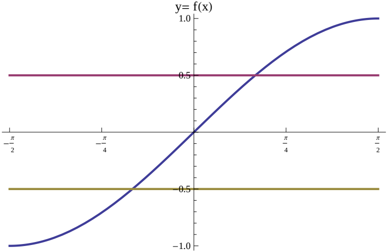
The trigonometric function $\sin x$ is not one-to-one functions, hence in order to create an inverse, we must restrict its domain.

The restricted sine function is given by

$$f(x) = \begin{cases} \sin x & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ \text{undefined} & \text{otherwise} \end{cases}$$

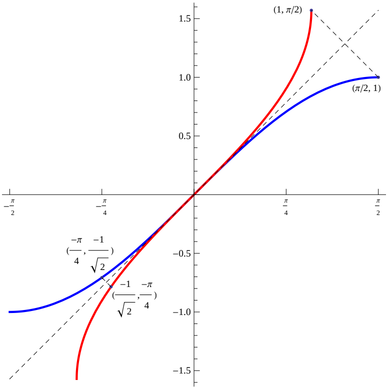
We have Domain(f) $= [-\frac{\pi}{2}, \frac{\pi}{2}]$ and Range(f) $= [-1, 1]$.





Inverse Sine Function (arcsin x $= sin^{-1}x$).

We see from the graph of the restricted sine function (or from its derivative) that the function is one-to-one and hence has an inverse, shown in red in the diagram below.



This inverse function, $f^{-1}(x)$, is denoted by $\boxed{f^{-1}(x) = \sin^{-1} x \quad \text{or} \quad \arcsin x.}$

Properties of $\sin^{-1} x$.

$$\text{Domain}(\sin^{-1}) = [-1, 1] \text{ and } \text{Range}(\sin^{-1}) = [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$

Since $f^{-1}(x) = y$ if and only if $f(y) = x$, we have:

$$\boxed{\sin^{-1} x = y \text{ if and only if } \sin(y) = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.}$$

Since $f(f^{-1})(x) = x \quad f^{-1}(f(x)) = x$ we have:

$$\boxed{\sin(\sin^{-1}(x)) = x \text{ for } x \in [-1, 1] \quad \sin^{-1}(\sin(x)) = x \text{ for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].}$$

from the graph: $\sin^{-1} x$ is an odd function and $\sin^{-1}(-x) = -\sin^{-1} x$.

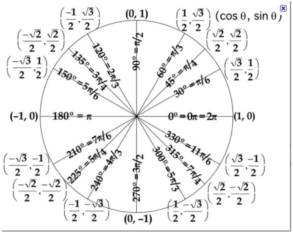
Evaluating $\sin^{-1} x$.

Example Evaluate $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ using the graph above.

- *We see that the point* $\left(\frac{-1}{\sqrt{2}}, \frac{-\pi}{4}\right)$ *is on the graph of* $y = \sin^{-1} x$*.*
- *Therefore* $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{-\pi}{4}$.

Example Evaluate $\sin^{-1}(\sqrt{3}/2)$ and $\sin^{-1}(-\sqrt{3}/2)$.

- $\sin^{-1}(\sqrt{3}/2) = y$ *is the same statement as:*
  y *is an angle between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ *with* $\sin y = \sqrt{3}/2$*.*
- *Consulting our unit circle, we see that* $y = \frac{\pi}{3}$*.*



- $\sin^{-1}(-\sqrt{3}/2) = -\sin^{-1}(\sqrt{3}/2) = -\frac{\pi}{3}$

More Examples For $\sin^{-1} x$

Example Evaluate $\sin^{-1}(\sin\pi)$.

- *We have* $\sin\pi = 0$, *hence* $\sin^{-1}(\sin\pi) = \sin^{-1}(0) = 0$.

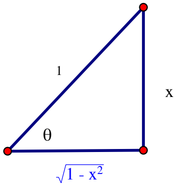
Example Evaluate $\cos(\sin^{-1}(\sqrt{3}/2))$.

- *We saw above that* $\sin^{-1}(\sqrt{3}/2) = \frac{\pi}{3}$.
- *Therefore* $\cos(\sin^{-1}(\sqrt{3}/2)) = \cos\left(\frac{\pi}{3}\right) = 1/2$.

Preparation for the method of Trigonometric Substitution

Example Give a formula in terms of x for $\tan(\sin^{-1}(x))$

- *We draw a right angled triangle with* $\theta = \sin^{-1} x$.



- *From this we see that* $\tan(\sin^{-1}(x)) = \tan(\theta) = \frac{x}{\sqrt{1-x^2}}$.

Derivative of $\sin^{-1} x$.

$$\boxed{\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad -1 \le x \le 1.}$$

Please read through the proof given in your notes using implicit differentiation. We can also derive a formula for $\frac{d}{dx}\sin^{-1}(k(x))$ using the chain rule, or we can apply the above formula along with the chain rule directly.

ExampleFind the derivative

$$\frac{d}{dx}\sin^{-1}\sqrt{\cos x}$$

.

- *We have* $\frac{d}{dx}\sin^{-1}\sqrt{\cos x} = \frac{1}{\sqrt{1-(\sqrt{\cos x})^2}}\frac{d}{dx}\sqrt{\cos x}$
- $$= \frac{1}{\sqrt{1-\cos x}} \cdot \frac{-\sin x}{2\sqrt{\cos x}} = \frac{-\sin x}{2\sqrt{\cos x}\sqrt{1-\cos x}}$$

Inverse Cosine Function

Inverse Cosine Function We can define the function $\cos^{-1} x = \arccos(x)$ similarly. The details are given at the end of your lecture notes.

$$\text{Domain}(\cos^{-1}) = [-1, 1] \quad \text{and} \quad \text{Range}(\cos^{-1}) = [0, \pi].$$

$$\boxed{\cos^{-1} x = y \quad \text{if and only if} \quad \cos(y) = x \ \text{ and } \ 0 \le y \le \pi.}$$

$$\boxed{\cos(\cos^{-1}(x)) = x \text{ for } x \in [-1, 1] \quad \cos^{-1}(\cos(x)) = x \text{ for } x \in \left[0, \pi\right].}$$

It is shown at the end of the lecture notes that

$$\frac{d}{dx}\cos^{-1} x = -\frac{d}{dx}\sin^{-1} x = \frac{-1}{\sqrt{1 - x^2}}$$

and one can use this to prove that

$$\boxed{\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}}.$$

Restricted Tangent Function

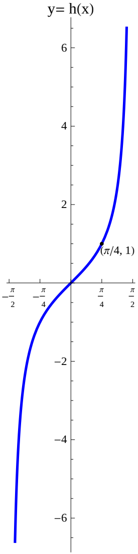
The tangent function is not a one to one function.
The restricted tangent function is given by

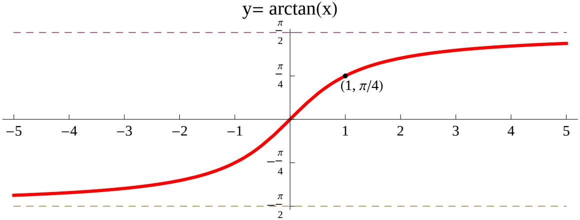
$$h(x) = \begin{cases} \tan x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \text{undefined} & \text{otherwise} \end{cases}$$

We see from the graph of the restricted tangent function (or from its derivative) that the function is one-to-one and hence has an inverse, which we denote by

$$\boxed{h^{-1}(x) = \tan^{-1} x \quad \text{or} \quad \arctan x.}$$

Graphs of Restricted Tangent and $tan^{-1}x$.





Properties of $tan^{-1}x$.

$$\text{Domain}(\tan^{-1}) = (-\infty, \infty) \text{ and } \text{Range}(\tan^{-1}) = (-\tfrac{\pi}{2}, \tfrac{\pi}{2}).$$

Since $h^{-1}(x) = y$ if and only if $h(y) = x$, we have:

$$\boxed{\tan^{-1} x = y \text{ if and only if } \tan(y) = x \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}.}$$

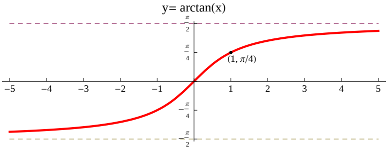
Since $h(h^{-1}(x)) = x$ and $h^{-1}(h(x)) = x$, we have:

$$\boxed{\tan(\tan^{-1}(x)) = x \text{ for } x \in (-\infty, \infty) \quad \tan^{-1}(\tan(x)) = x \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).}$$

From the graph, we have: $\boxed{\tan^{-1}(-x) = -\tan^{-1}(x).}$

Also, since $\lim_{x \to (\frac{\pi}{2}-)} \tan x = \infty$ and $\lim_{x \to (-\frac{\pi}{2}+)} \tan x = -\infty$,

we have $\boxed{\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}}$ and $\boxed{\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}}$



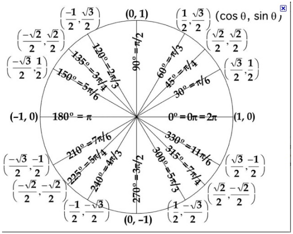
Evaluating $\tan^{-1} x$

Example Find $\tan^{-1}(1)$ and $\tan^{-1}(\frac{1}{\sqrt{3}})$.

- $\tan^{-1}(1)$ *is the unique angle,* θ*, between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ *with* $\tan\theta = \frac{\sin\theta}{\cos\theta} = 1$*. By inspecting the unit circle, we see that* $\theta = \frac{\pi}{4}$*.*
- $\tan^{-1}(\frac{1}{\sqrt{3}})$ *is the unique angle,* θ*, between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ *with* $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{3}}$*. By inspecting the unit circle, we see that* $\theta = \frac{\pi}{6}$*.*

Example Find $\cos(\tan^{-1}(\sqrt{3}))$.

- $\cos(\tan^{-1}(\sqrt{3})) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$.



Derivative of $\tan^{-1} x$.

Using implicit differentiation, we get

$$\boxed{\frac{d}{dx}\tan^{-1} x = \frac{1}{x^2+1}, \quad -\infty < x < \infty.}$$

(Please read through the proof in your notes.) We can use the chain rule in conjunction with the above derivative.

Example Find the domain and derivative of $\tan^{-1}(\ln x)$

- *Domain = Domain of* $\ln x = (0, \infty)$
- $$\frac{d}{dx}\tan^{-1}(\ln x) = \frac{\frac{1}{x}}{1+(\ln x)^2} = \frac{1}{x(1+(\ln x)^2)}.$$

Integration Formulas

Reversing the derivative formulas above, we get

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C, \quad \int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C,$$

Example

$$\int_0^{1/2} \frac{1}{1+4x^2}\,dx$$

- *We use substitution. Let* $u = 2x$*, then* $du = 2dx$*,* $\quad u(0) = 0$*,* $u(1/2) = 1$*.*
- $$\int_0^{1/2} \frac{1}{1+4x^2}\,dx = \frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1} u\Big|_0^1 = \frac{1}{2}[\tan^{-1}(1) - \tan^{-1}(0)]$$
- $$\frac{1}{2}[\frac{\pi}{4} - 0] = \frac{\pi}{8}.$$

Integration

Example

$$\int \frac{1}{\sqrt{9-x^2}}\,dx$$

- $$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{3\sqrt{1-\frac{x^2}{9}}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{1-\frac{x^2}{9}}}dx$$
- *Let* $u = \frac{x}{3}$, *then* $dx = 3du$
- $$\int \frac{1}{\sqrt{9-x^2}}\,dx = \frac{1}{3}\int \frac{3}{\sqrt{1-u^2}}du = \sin^{-1} u + C = \sin^{-1}\frac{x}{3} + C$$